

# REPRESENTATIONS OF BICIRCULAR LIFT MATROIDS

RONG CHEN AND ZIFEI GAO

**ABSTRACT.** Bicircular lift matroids are a class of matroids defined on the edge set of a graph. For a given graph  $G$ , the circuits of its bicircular lift matroid are the edge sets of those subgraphs of  $G$  that contain at least two cycles, and are minimal with respect to this property. The main result of this paper is a characterization of when two graphs give rise to the same bicircular lift matroid, which answers a question proposed by Irene Pivotto. In particular, aside from some appropriately defined “small” graphs, two graphs have the same bicircular lift matroid if and only if they are 2-isomorphic in the sense of Whitney.

*Key Words:* bicircular lift matroids, representation.

## 1. INTRODUCTION

We assume the reader is familiar with fundamental definitions in matroid and graph theory. For a graph  $G$ , a set  $X \subseteq E(G)$  is a *cycle* if  $G[X]$  is a connected 2-regular graph. Bicircular lift matroids are a class of matroids defined on the edge set of a graph. For a given graph  $G$ , the circuits of its bicircular lift matroid  $L(G)$  are the edge sets of those subgraphs of  $G$  that contain at least two cycles, and are minimal with respect to this property. That is, the circuits of  $L(G)$  consists of the edge sets of two edge-disjoint cycles with at most one common vertex, or three internally disjoint paths between a pair of distinct vertices. Bicircular lift matroids are a special class of lift matroids that arises from biased graphs. Biased graphs and lift matroids were introduced by Zaslavsky in [8, 9].

Whitney [6] characterized which graphs have isomorphic graphic matroids. Chen, DeVos, Funk and Pivotto [2] generalized Whitney’s result and characterized which biased graphs have isomorphic graphic frame matroids. Matthews [4] characterized which graphs give rise to isomorphic bicircular matroids that are graphic. Coullard, del Greco and Wagner [3, 7] characterized which graphs give rise to isomorphic bicircular matroids. In this paper, we characterize which graphs give rise to isomorphic bicircular lift matroids, which answers a question proposed by Pivotto in the Matroid Union blog [5]. In particular, except for some special graphs, each of which is a subdivision of a graph on at most four vertices, two graphs have the same bicircular lift matroid if and only if they are 2-isomorphic in the sense of Whitney [6]. The main result is used in [1] to prove that the class of matroids that are graphic or bicircular lift has a finite list of excluded minors.

To state our result completely we need more definitions. Let  $k, l, m$  be positive integers. We denote by  $K_m$  the complete graph with  $m$  vertices. We denote by  $K_2^m$  the graph obtained from  $K_2$  with its unique edge replaced by  $m$  parallel edges. And we denote by  $K_3^{k,l,m}$  the graph obtained from  $K_3$  with its three edges replaced by  $k, l, m$  parallel edges respectively. A graph obtained from graph  $G$  by replacing some edges of  $G$  with internally

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disjoint paths is a *subdivision* of  $G$ . Note that  $G$  is a subdivision of itself. A path  $P$  of a connected graph  $G$  is an *ear* if each internal vertex of  $P$  has degree two and each end-vertex has degree at least three in  $G$ , and  $P$  is contained in a cycle. A graph  $G$  is *2-edge-connected* if each edge of  $G$  is contained in some cycle. Let  $M(G)$  denote the graphic matroid of a graph  $G$ .

Given a set  $X$  of edges, we let  $G|X$  denote the subgraph of  $G$  with edge set  $X$  and no isolated vertices. Let  $(X_1, X_2)$  be a partition of  $E(G)$  such that  $V(G|X_1) \cap V(G|X_2) = \{u_1, u_2\}$ . We say that  $G'$  is obtained by a *Whitney Switching* on  $G$  on  $\{u_1, u_2\}$  if  $G'$  is a graph obtained by identifying vertices  $u_1, u_2$  of  $G|X_1$  with vertices  $u_2, u_1$  of  $G|X_2$ , respectively. A graph  $G'$  is *2-isomorphic* to  $G$  if  $G'$  is obtained from  $G$  by a sequence of the operations: Whitney switchings, identifying two vertices from distinct components of a graph, or partitioning a graph into components each of which is a block of the original graph.

**Theorem 1.1.** (*Whitney's 2-Isomorphism Theorem*) *Let  $G_1$  and  $G_2$  be graphs. Then  $M(G_1) \cong M(G_2)$  if and only if  $G_1$  and  $G_2$  are 2-isomorphic.*

It follows from Theorem 1.1 that if  $G_1$  and  $G_2$  are 2-isomorphic, then  $L(G_1) = L(G_2)$ . The converse, however, is not true. This can be seen by choosing  $G_1$  and  $G_2$  to be isomorphic to  $K_4$ , but not to each other. Much of the remainder of the paper is aimed at characterizing when the converse to this statement is not true.

Let  $G_1$  and  $G_2$  be graphs with  $L(G_1) = L(G_2)$ . Since  $E(G_i)$  is independent in  $L(G_i)$  if and only if  $G_i$  has at most one cycle, we may assume that  $G_1$  and  $G_2$  have at least two cycles. Moreover, since  $e$  is a cut-edge of  $G_1$  if and only if  $e$  is a cut-edge of  $G_2$  or  $G_2 \setminus e$  is a forest, an edge is a cut-edge of  $G_1$  if and only if it is a cut-edge of  $G_2$ . Hence, to simplify the analysis below, it will be assumed for the remainder of the paper that  $G_1$  and  $G_2$  are 2-edge-connected. Observe that when  $L(G_1)$  has only one circuit, it is straightforward to characterize the structure of both  $G_1$  and  $G_2$ . Thus, the remainder of the paper will further restrict the analysis to the case that  $L(G_1)$  has at least two circuits. In the paper, we prove

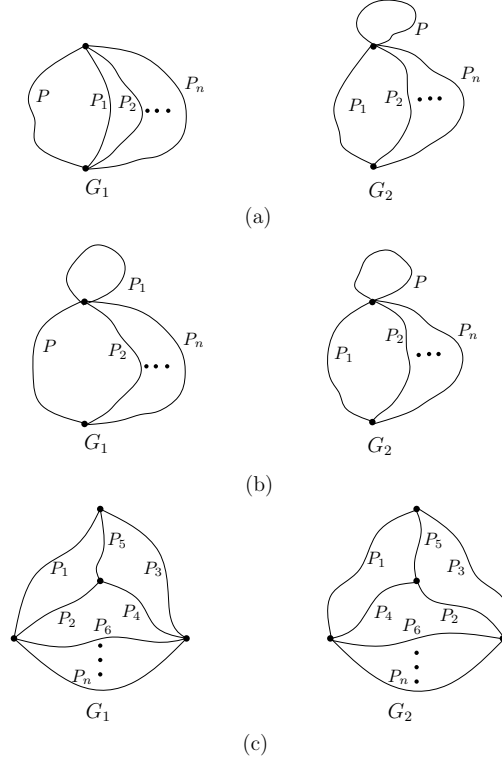
**Theorem 1.2.** *Let  $G_1$  be a 2-edge-connected graph such that  $L(G_1)$  contains at least two circuits. Let  $G_2$  be a graph with  $L(G_1) = L(G_2)$ . Then at least one of the following holds.*

- (1)  $G_1$  and  $G_2$  are 2-isomorphic.
- (2)  $G_1$  and  $G_2$  are 2-isomorphic to subdivisions of  $K_4$ , where the edge set of an ear of  $G_1$  is also the edge set of an ear of  $G_2$ .
- (3)  $G_1$  and  $G_2$  are 2-isomorphic to subdivisions of  $K_3^{m,2,n}$  for some  $m \in \{1, 2\}$  and  $n \geq 2$ , where the edge set of an ear of  $G_1$  is also the edge set of an ear of  $G_2$ . Moreover, when  $n \geq 3$ , the  $n$  ears in  $G_1$  having the same ends also have the same ends in  $G_2$ .
- (4)  $G_1$  and  $G_2$  are 2-isomorphic to the graphs pictured in Figure 1.

The following result, which is an easy consequence of Theorem 1.2, is used in [1] to prove that the class of matroids that are graphic or bicircular lift matroids has a finite list of excluded minors.

Two elements are a *series pair* of a graph  $G$  if and only if each cycle can not intersect them in exactly one element. A *series class* is a maximal set  $X \subseteq E(G)$  such that every two edges of  $X$  form a series pair. Let  $\text{co}(G)$  denote a graph obtained from  $G$  by contracting all cut-edges from  $G$  and then, for each series class  $X$ , contracting all but one distinguished element of  $X$ .

**Corollary 1.3.** *Let  $G_1$  and  $G_2$  be connected graphs with  $L(G_1) = L(G_2)$  and such that  $L(G_1)$  has at least two circuits. If  $|V(\text{co}(G_1))| \geq 5$  then  $G_1$  and  $G_2$  are 2-isomorphic.*


 FIGURE 1. In (a) and (b),  $n \geq 3$ ; and in (c),  $n \geq 2$ 

## 2. PROOF OF THEOREM 1.2.

Let  $G$  be a graph, and  $e, f \in E(G)$ . We say that  $e$  is a *link* if it has distinct end-vertices; otherwise  $e$  is a *loop*. If  $\{e, f\}$  is a cycle, then  $e$  and  $f$  are *parallel*. A *parallel class* of  $G$  is a maximal subset  $P$  of  $E(G)$  such that any two members of  $P$  are parallel and no member is a loop. Moreover, if  $|P| \geq 2$  then  $P$  is *non-trivial*; otherwise  $P$  is *trivial*. Let  $\text{si}(G)$  denote the graph obtained from  $G$  by deleting all loops and all but one distinguished element of each non-trivial parallel class. Obviously, the graph we obtain is uniquely determined up to a renaming of the distinguished elements. If  $G$  has no loops and no non-trivial parallel class, then  $G$  is *simple*.

The following result is implied in ([9], Theorem 3.6.).

**Lemma 2.1.** *Let  $e$  be an edge of a graph  $G$ . Then we have*

- (1)  $L(G \setminus e) = L(G) \setminus e$ ;
- (2) *when  $e$  is a loop,  $L(G)/e = M(G \setminus e)$ ;*
- (3) *when  $e$  is a link,  $L(G)/e = L(G/e)$ .*

Corollary 2.2 follows immediately from Lemma 2.1 (2) and Theorem 1.1.

**Corollary 2.2.** *Let  $G_1, G_2$  be graphs with  $L(G_1) = L(G_2)$ , and  $e$  a loop of both  $G_1$  and  $G_2$ . Then  $G_1$  and  $G_2$  are 2-isomorphic.*

The idea used to prove the following Lemma was given by the referee.

**Lemma 2.3.** *Let  $G_1$  and  $G_2$  be connected graphs without loops and with  $|V(G_1)| = |V(G_2)|$  and  $E(G_1) = E(G_2)$ . Assume that for each edge  $e \in E(G_1)$  the graphs  $G_1/e$  and  $G_2/e$  are 2-isomorphic. Then  $G_1$  and  $G_2$  are 2-isomorphic.*

*Proof.* By Whitney's 2-Isomorphism Theorem, to prove the result it suffices to show that each spanning tree of  $G_1$  is also a spanning tree of  $G_2$ . Let  $T_1$  be a spanning tree of  $G_1$ , and let  $T_2$  be the subgraph of  $G_2$  induced by  $E(T_1)$ . Assume that  $T_2$  is not a spanning tree of  $G_2$ . Since  $|V(G_1)| = |V(G_2)|$ , the subgraph  $T_2$  contains a cycle  $C$ . Let  $e$  be an edge in  $E(T_1)$ . Then  $T_1/e$  is acyclic and  $T_2/e$  is not, and so  $G_1/e$  and  $G_2/e$  are not 2-isomorphic; a contradiction.  $\square$

**Lemma 2.4.** *Let  $G_1$  be a 2-edge-connected graph such that  $L(G_1)$  contains at least two circuits. Let  $G_2$  be a graph with  $L(G_1) = L(G_2)$ . Assume that  $G_1$  has a link  $e$  such that  $e$  is a loop of  $G_2$ . Then  $G_1$  and  $G_2$  are 2-isomorphic to the graphs pictured in Figure 2.*

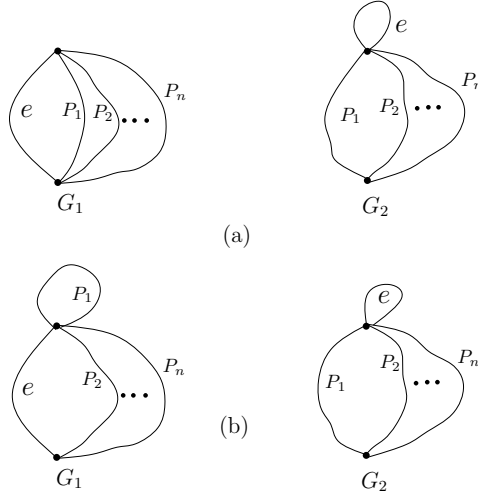
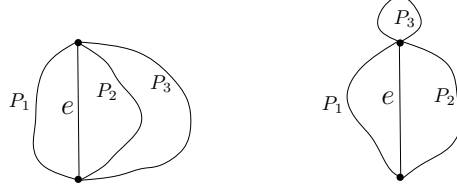


FIGURE 2.  $n \geq 3$

*Proof.* Since  $L(G_1)$  contains at least two circuits and  $L(G_1) = L(G_2)$ , the graph  $G_2 - \{e\}$  has cycles  $C_1$  and  $C_2$  such that  $C_1 \cup C_2$  is a circuit of  $L(G_2)$ . Since  $e$  is a loop of  $G_2$ , for some integer  $k \in \{2, 3\}$  there is a partition  $(P_1, P_2, \dots, P_k)$  of  $E(C_1 \cup C_2)$  such that when  $k = 2$  the sets  $P_1 \cup \{e\}$  and  $P_2 \cup \{e\}$  are circuits of  $L(G_1)$ , and when  $k = 3$  the sets  $P_1 \cup P_2 \cup \{e\}$ ,  $P_2 \cup P_3 \cup \{e\}$  and  $P_1 \cup P_3 \cup \{e\}$  are circuits of  $L(G_1)$ . Since  $E(C_1 \cup C_2)$  is also a circuit of  $L(G_1)$  and  $e$  is a link of  $G_1$ , it is easy to verify that  $k = 3$  (that is,  $C_1 \cup C_2$  is a theta-subgraph of  $G_2$ .) and (1)  $G_1|C_1 \cup C_2 \cup \{e\}$  is 2-isomorphic to graphs pictured in Figure 3. Hence, by the arbitrary choice of  $C_1$  and  $C_2$ , (2) no two cycles in  $G_2$  have at most one common vertex; and (3) each ear of a theta-subgraph of  $G_2$  is a cycle in  $G_1$  or a path connecting the end-vertices of  $e$  in  $G_1$ .

For each edge  $f \in E(G_2) - (C_1 \cup C_2 \cup \{e\})$ , there is a set  $X$  with  $f \in X \subseteq E(G_2) - (C_1 \cup C_2 \cup \{e\})$  such that  $G_2|C_1 \cup C_2 \cup X$  is 2-edge-connected. By (2)  $G_2|C_1 \cup C_2 \cup X$  is a subdivision of  $K_4$  or  $K_2^4$ . (1) and (3) imply that  $G_2|C_1 \cup C_2 \cup X$  is a subdivision of  $K_2^4$ . Repeating the process several times, we have that  $G_2 - \{e\}$  is a  $K_2^n$ -subdivision for some integer  $n \geq 3$ . Hence,  $G_1$  and  $G_2$  are 2-isomorphic to the graphs pictured in Figure 2.  $\square$


 FIGURE 3. Possible structures of graph  $G_1|C_1 \cup C_2 \cup \{e\}$ .

By Lemma 2.4, to prove Theorem 1.2 we only need to consider the case that an edge is a link in  $G_1$  if and only if it is a link in  $G_2$ .

**Lemma 2.5.** *Let  $G_1$  and  $G_2$  be connected and 2-edge-connected graphs with  $L(G_1) = L(G_2)$  such that  $L(G_1)$  has at least two circuits and such that each series class of  $G_i$  is an ear of  $G_i$  for each  $i \in \{1, 2\}$ . Then a set of edges is the edge set of an ear of  $G_1$  if and only if it is the edge set of an ear of  $G_2$ .*

*Proof.* Assume otherwise. Without loss of generality assume that  $e$  and  $f$  are contained in some ear of  $G_1$ , but not in the same ear of  $G_2$ . Evidently,  $e$  is not in any cycle of  $G_1 - \{f\}$  and  $L(G_1 - \{f\})$  has a circuit as  $L(G_1)$  has at least two circuits. Moreover, since  $L(G_1 - \{f\}) = L(G_2 - \{f\})$ , the edge  $e$  is a coloop of  $G_2 - \{f\}$ ; so  $\{e, f\}$  is a bond of  $G_2$ . Then  $e$  and  $f$  are contained in the same ear of  $G_2$  as each series class of  $G_2$  is an ear of  $G_2$ , a contradiction.  $\square$

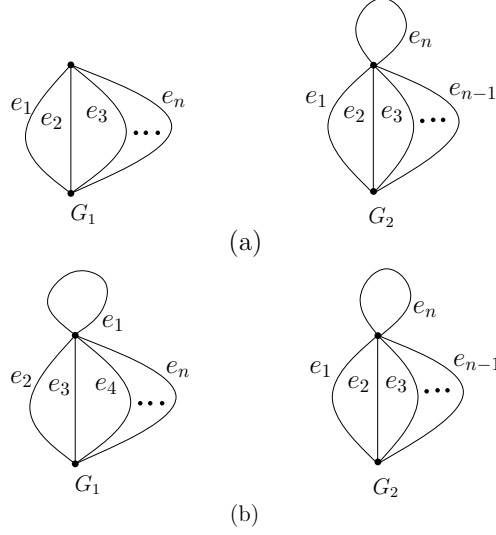
By possibly applying a sequence of Whitney's switching we can assume that each series class in a graph  $G$  is an ear of  $G$ . Furthermore, by Lemma 2.5 we can further assume that a set of edges is the edge set of an ear of  $G_1$  if and only if it is the edge set of an ear of  $G_2$ . Hence, we only need consider cosimple graphs, where a graph is *cosimple* if it has no cut-edges or non-trivial series classes.

Let  $\text{loop}(G)$  be the set consisting of loops of  $G$ .

**Lemma 2.6.** *Let  $G_1$  and  $G_2$  be cosimple 2-edge-connected graphs with  $2 \leq |V(G_1)| = |V(G_2)| \leq 3$ . Assume that  $L(G_1) = L(G_2)$  and  $L(G_1)$  contains at least two circuits. Then exactly one of the following holds.*

- (1)  $G_1$  and  $G_2$  are 2-isomorphic.
- (2)  $|V(G_1)| = 2$ , the graphs  $G_1$  and  $G_2$  are isomorphic to the graphs pictured in Figure 4.
- (3)  $G_1$  and  $G_2$  are 2-isomorphic to  $K_3^{m,2,n}$  for some integers  $m \in \{1, 2\}$  and  $n \geq 2$ , moreover, the  $n$  parallel edges in  $G_1$  are also the  $n$  parallel edges in  $G_2$  when  $n \geq 3$ .

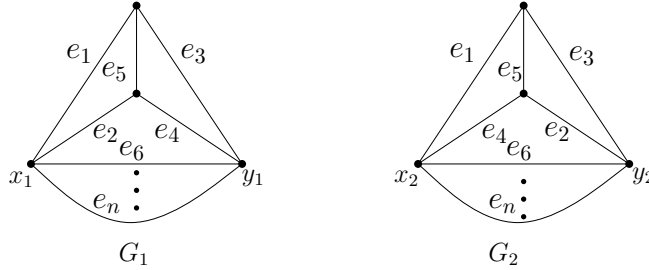
*Proof.* By Lemma 2.4 we may assume that  $\text{loop}(G_1) = \text{loop}(G_2)$ . Then the lemma holds when  $|V(G_1)| = 2$ . So assume that  $|V(G_1)| = 3$ . Since  $\text{loop}(G_1) = \text{loop}(G_2)$ , each non-trivial parallel class of  $G_1$  with at least three edges must be also a non-trivial parallel class of  $G_2$ . Hence, when  $G_1$  has two parallel classes with at least three edges, (1) holds. So we may assume that  $G_1$  has at most one parallel class with at least three edges. On the other hand, since  $G_1$  and  $G_2$  are cosimple,  $G_1$  and  $G_2$  have three parallel classes and at least two of them are non-trivial. Hence, when  $G_1$  has no loops, (3) obviously holds; when  $G_1$  has a loop, since  $\text{loop}(G_1) = \text{loop}(G_2)$ , Corollary 2.2 implies that  $G_1$  and  $G_2$  are 2-isomorphic, that is, (1) holds.  $\square$

FIGURE 4.  $n \geq 3$ .

The star of a vertex  $v$  in a graph  $G$ , denoted by  $st_G(v)$ , is the set of edges of  $G$  incident with  $v$ .

**Lemma 2.7.** *Let  $G_1$  and  $G_2$  be 2-edge-connected cosimple graphs with exactly four vertices and without loops. Assume that  $L(G_1) = L(G_2)$  and  $L(G_1)$  has at least two circuits. Then at least one of the following holds.*

- (1)  $G_1$  and  $G_2$  are 2-isomorphic;
- (2)  $G_1$  and  $G_2$  are isomorphic to  $K_4$ ;
- (3)  $G_1$  and  $G_2$  are 2-isomorphic to the graphs pictured in Figure 5.

FIGURE 5.  $n \geq 7$ .

*Proof.* By Lemma 2.1 (3), for each edge  $e \in E(G_1)$  we have  $L(G_1/e) = L(G_2/e)$ . If  $G_1/e$  and  $G_2/e$  are 2-isomorphic for each edge  $e \in E(G_1)$ , then Lemma 2.3 implies that  $G_1$  and  $G_2$  are 2-isomorphic. So we may assume that there is some edge  $f \in E(G_1)$  such that  $G_1/f$  and  $G_2/f$  are not 2-isomorphic. Since  $L(G_1)$  has at least two circuits,  $L(G_1/f)$  also has at least two circuits. Moreover, since  $G_1/f$  and  $G_2/f$  are cosimple graphs with

exactly three vertices, by Lemma 2.6 we have that (a) the graphs  $G_1/f$  and  $G_2/f$  are isomorphic to  $K_3^{m,2,n}$  for some integers  $m \in \{1, 2\}$  and  $n \geq 2$ ; moreover, when  $n \geq 3$  the  $n$  parallel edges in  $G_1$  are also the  $n$  parallel edges in  $G_2$ .

**2.7.1.** *Two edges are parallel in  $G_1$  if and only if they are parallel in  $G_2$ .*

*Subproof.* If two edges are parallel in  $G_1$  but not  $G_2$ , then contracting one of the edges produces a counterexample to Lemma 2.4.  $\square$

The simple proof of 2.7.1 is given by the referee. Since no non-trivial parallel classes in  $G_1$  or  $G_2$  contains  $f$  by (a), 2.7.1 implies

**2.7.2.** *Each 2-edge path joining the end-vertices of a non-trivial parallel class of  $G_1$  is also a 2-edge path joining the end-vertices of the non-trivial parallel class of  $G_2$ .*

**2.7.3.** *Let  $P_1, P_2$  be non-trivial parallel classes of  $G_1$ . Then  $\text{si}(G_1|P_1 \cup P_2 \cup f)$  is a triangle.*

*Subproof.* Since  $G_1/f$  has no loop, neither  $P_1$  nor  $P_2$  contains  $f$ . If  $P_1$  and  $P_2$  are not contained in a parallel class of  $G_1/f$ , then  $P_1$  and  $P_2$  are contained in two different non-trivial parallel classes of  $G_1/f$ . Moreover, since  $P_1$  and  $P_2$  are also non-trivial parallel classes of  $G_2$  by 2.7.1, by (a) we have that  $G_1/f$  and  $G_2/f$  are isomorphic, a contradiction. So  $P_1$  and  $P_2$  are contained in a parallel class of  $G_1/f$ . Then  $\text{si}(G_1|P_1 \cup P_2 \cup f)$  is a triangle.  $\square$

First we consider the case that  $G_1/f$  is isomorphic to  $K_3^{2,2,n}$ . By 2.7.3,  $G_1$  is obtained from  $G_1/f$  by splitting a degree-4 vertex. Since  $G_1$  is cosimple,  $G_1$  is isomorphic to the graph pictured in Figure 5 with  $e_5$  relabelled by  $f$ . Let  $P$  be the unique non-trivial parallel class of  $G_1$  with  $n$  edges. Since  $P$  is also non-trivial parallel class of  $G_2$  by 2.7.1 and the fact that  $G_2/f$  is isomorphic to  $K_3^{2,2,n}$ , the graph  $G_2$  is isomorphic to the graph pictured in Figure 5 with  $e_5$  relabelled by  $f$ . So (3) holds.

Secondly we consider the case that  $G_1/f$  is isomorphic to  $K_3^{1,2,n}$ . Let  $e_i$  be the edge of  $G_i/f$  that is not in a parallel class for  $1 \leq i \leq 2$ . Evidently, when  $n \geq 3$ , since  $G_1/f$  and  $G_2/f$  are not 2-isomorphic,  $e_1 \neq e_2$ . Since each vertex of  $G_1$  has degree at least three, by 2.7.3 the graph  $G_1$  is obtained from  $G_1/f$  by splitting the vertex  $v$  incident with two non-trivial parallel classes. When  $|st_{G_1/f}(v)| = 4$ , since  $G_1$  is cosimple  $G_1$  is isomorphic to  $K_4$ . By symmetry  $G_2$  is also isomorphic to  $K_4$ . So (2) holds.

Assume that  $|st_{G_1/f}(v)| \geq 5$ , that is, a non-trivial parallel class  $P$  incident with  $v$  in  $G_1/f$  has at least three edges. Then some proper subset  $P'$  of  $P$  is a non-trivial parallel class in  $G_1$  as  $G_1$  is cosimple. Let  $\{f_1, f_2\}$  be the 2-edge parallel class in  $G_1/f$ . Since  $\{f, f_1, f_2\}$  is a cycle in  $G_1$  and  $\{e_1, f_1, f_2\}$  is the neighbourhood of a degree-3 vertex in  $G_1/f$  and  $G_1$ , by symmetry we may assume that  $e_1, f_1$  is a 2-edge path joining the end-vertices of  $P'$  in  $G_1$  and  $f_2$  is not incident with  $P'$ . On the other hand, by symmetry,  $e_2$  is also contained in a 2-edge path joining the end-vertices of  $P'$  in  $G_2$ . So  $f_1 = e_2$  as  $e_2 \in \{f_1, f_2\}$ , consequently,  $|P - P'| = 1$ , for otherwise there are two such  $P'$ , which is not possible. Therefore, (3) holds.  $\square$

**Lemma 2.8.** *Let  $G_1$  and  $G_2$  be 2-edge-connected cosimple graphs with five vertices and without loops. Assume that  $L(G_1) = L(G_2)$  and  $L(G_1)$  has at least two circuits. Then  $G_1$  and  $G_2$  are 2-isomorphic.*

*Proof.* By Lemma 2.1 (3), for each edge  $e \in E(G_1)$  we have  $L(G_1/e) = L(G_2/e)$ . If  $G_1/e$  and  $G_2/e$  are 2-isomorphic for each edge  $e \in E(G_1)$ , then Lemma 2.3 implies that  $G_1$  and  $G_2$  are 2-isomorphic. So we may assume that for some edge  $f \in E(G_1)$  we have  $L(G_1/f) = L(G_2/f)$  but  $G_1/f$  and  $G_2/f$  are not 2-isomorphic.

We claim that  $G_1/f$  and  $G_2/f$  have no loops. Since  $L(G_1/f)$  has at least two circuits, Lemma 2.4 implies that  $\text{loop}(G_1/f) = \text{loop}(G_2/f)$ . If  $\text{loop}(G_1/f) \neq \emptyset$ , then Corollary 2.2 implies that  $G_1/f$  and  $G_2/f$  are 2-isomorphic, a contradiction.

Since  $G_1/f$  and  $G_2/f$  are cosimple with four vertices and without loops, Lemma 2.7 implies that  $G_1/f$  and  $G_2/f$  are either 2-isomorphic to  $K_4$  or to the graphs pictured in Figure 5. Since each vertex in  $K_4$  has degree three and  $G_1$  and  $G_2$  are cosimple, neither  $G_1/f$  nor  $G_2/f$  is 2-isomorphic to  $K_4$ . So  $G_1/f$  and  $G_2/f$  are 2-isomorphic to the graphs pictured in Figure 5 with  $G_i$  replaced by  $G_i/f$  and all other labeling the same. Let  $P$  be the non-trivial parallel class in  $G_1/f$  and  $G_2/f$ . For each  $i \in \{1, 2\}$ , let  $u_i$  and  $v_i$  be the end-vertices of  $f$  in  $G_i$ , let  $x_i$  be the vertex of degree at least four in  $G_i/f$  incident with  $e_1$ , and  $y_i$  be the vertex of degree at least four in  $G_i/f$  incident with  $e_3$ . Since  $|\text{st}_{G_i}(u_i)|, |\text{st}_{G_i}(v_i)| \geq 3$ , the graph  $G_i$  is obtained from  $G_i/f$  by splitting  $x_i$  or  $y_i$ . Without loss of generality we may assume that  $G_i$  is obtained from  $G_i/f$  by splitting  $x_i$  for each  $i \in \{1, 2\}$ .

We claim that  $|E(G_1/f)| = 7$ , that is,  $|P| = 2$ . Assume otherwise. Then there is a subset  $P'$  of  $P$  with  $|P'| \geq 2$  such that  $P'$  is also a parallel class in  $G_1$ . Using a similar analysis to the one in the proof of 2.7.1 we have that  $P'$  is also a parallel class in  $G_2$ . Assume that  $e_1, e_2$  are adjacent in  $G_1$ . Since a union of any two edges in  $P'$  and  $\{e_1, e_2, e_5\}$  or  $\{e_3, e_4, e_5\}$  is a circuit of  $L(G_1)$ , we deduce that  $\{e_1, e_4, f\} \cup P'$  are contained in  $\text{st}_{G_2}(u_2)$  or  $\text{st}_{G_2}(v_2)$ . Hence,  $|P - P'| \leq 1$ , implying that  $(P - P') \cup \{f\}$  is a bond of  $G_2$  with at most two edges, a contradiction as  $G_2$  is cosimple. So  $e_1, e_2$  are not adjacent in  $G_1$ . By symmetry we may assume that  $\text{st}_{G_1}(v_1) = \{e_2, f\} \cup P'$ . Since  $P'$  is a parallel class of  $G_2$  and the union of  $\{e_3, e_4, e_5\}$  and any two edges in  $P'$  is a circuit of  $L(G_1)$ , by symmetry we may assume that  $\{e_4, f\} \cup P'$  are incident with  $v_2$ . Hence,  $|P - P'| = 1$  and  $\text{st}_{G_1}(u_1) = \text{st}_{G_2}(u_2) = (P - P') \cup \{e_1, f\}$ . Set  $\{e_6\} = P - P'$ . See Figure 6. Then  $\{e_1, e_2, e_3, e_5, e_6, f\}$  is a circuit of  $L(G_1)$  but is not a circuit of  $L(G_2)$ , a contradiction. So  $|E(G_1/f)| = 7$ . Set  $E(G_1/f) := \{e_1, e_2, \dots, e_7\}$ .

Since  $G_1$  and  $G_2$  are cosimple and  $|E(G_1/f)| = 7$ , we have  $|\text{st}_{G_i}(u_i)| = |\text{st}_{G_i}(v_i)| = 3$  for each  $i \in \{1, 2\}$ . By symmetry, there are two cases to consider. First we consider the case  $\text{st}_{G_1}(u_1) = \{f, e_1, e_2\}$ . Since  $\{e_1, e_2, e_3, e_4, e_5\}$  is a circuit of  $L(G_1)$ , by symmetry we can assume  $\text{st}_{G_2}(u_2) = \{f, e_1, e_4\}$ . Then  $\{e_2, e_3, e_5, e_6, e_7, f\}$  is a circuit of  $L(G_1)$  but is not a circuit of  $L(G_2)$ , a contradiction.

Secondly consider the case  $\text{st}_{G_1}(u_1) = \{f, e_1, e_6\}$ . Then  $\{e_1, e_3, e_4, e_5, e_6\}$  is a circuit of  $L(G_1)$ . On the other hand, by symmetry and the analysis in the last paragraph we have  $\{f, e_1, e_4\} \neq \{N_{G_2}(u_2), N_{G_2}(v_2)\}$ . So  $\{e_1, e_3, e_4, e_5, e_6\}$  is not a circuit of  $L(G_2)$ , a contradiction.  $\square$

For convenience, Theorem 1.2 is restated here.

**Theorem 2.9.** *Let  $G_1$  be a 2-edge-connected graph such that  $L(G_1)$  contains at least two circuits. Let  $G_2$  be a graph with  $L(G_1) = L(G_2)$ . Then at least one of the following holds.*

- (1)  $G_1$  and  $G_2$  are 2-isomorphic.
- (2)  $G_1$  and  $G_2$  are 2-isomorphic to subdivisions of  $K_4$ , where the edge set of an ear of  $G_1$  is also the edge set of an ear of  $G_2$ .



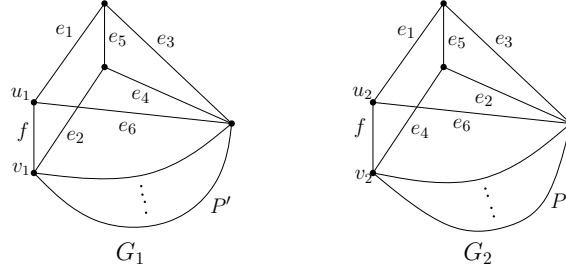


FIGURE 6.

- (3)  $G_1$  and  $G_2$  are 2-isomorphic to subdivisions of  $K_3^{m,2,n}$  for some  $m \in \{1, 2\}$  and  $n \geq 2$ , where the edge set of an ear of  $G_1$  is also the edge set of an ear of  $G_2$ . Moreover, when  $n \geq 3$ , the  $n$  ears in  $G_1$  having the same ends also have the same ends in  $G_2$ .
- (4)  $G_1$  and  $G_2$  are 2-isomorphic to the graphs pictured in Figure 1.

*Proof.* If some loop  $e$  of  $G_1$  is also a loop of  $G_2$ , then by Corollary 2.2 we have that  $G_1 \setminus e$  and  $G_2 \setminus e$  are 2-isomorphic. So  $G_1$  and  $G_2$  are 2-isomorphic. Moreover, when some link of  $G_1$  is a loop of  $G_2$ , Lemma 2.4 implies that (4) holds. Therefore, we may assume that neither  $G_1$  nor  $G_2$  has loops. By Whitney's 2-Isomorphism Theorem we can further assume that  $G_1$  and  $G_2$  are connected, and each series class of  $G_i$  is an ear of  $G_i$  for each  $i \in \{1, 2\}$ . Using Lemma 2.5 we may assume that a subset of  $E(G_1)$  is the edge set of an ear of  $G_1$  if and only if it is the edge set of an ear of  $G_2$ . Therefore, we may assume that  $G_1$  and  $G_2$  are cosimple.

Since the rank of  $L(G_i)$  is equal to  $|V(G_i)|$ , we have  $|V(G_1)| = |V(G_2)|$ . When  $|V(G_1)| \leq 4$ , Lemmas 2.6 and 2.7 imply that the result holds. We claim that when  $|V(G_1)| \geq 5$  we have that  $G_1$  and  $G_2$  are 2-isomorphic. When  $|V(G_1)| = 5$ , the claim follows from Lemma 2.8. So we may assume that  $|V(G_1)| \geq 6$ . For each edge  $e \in E(G_1)$ , by Lemma 2.1 (3) we have  $L(G_1/e) = L(G_2/e)$ . By induction  $G_1/e$  and  $G_2/e$  are 2-isomorphic. So  $G_1$  and  $G_2$  are 2-isomorphic by Lemma 2.3.  $\square$

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### REFERENCES

- [1] R. Chen. The excluded minors for the class of matroids that are graphic or bicircular lift. submitted.
- [2] R. Chen, M. DeVos, D. Funk, I. Pivotto. Graphical representations of graphic frame matroid. *Graphs. Combin.*, 31:2075-2086, 2015.
- [3] C. Coullard, J. del Greco, D. Wagner. Representations of bicircular matroids. *Discrete Appl. Math.*, 32: 223-240, 1991.
- [4] L.R. Matthews. Bicircular matroids. *Quart. J. Math. Oxford*, 28(2): 213-228, 1977.
- [5] I. Pivotto. <http://matroidunion.org/?p=279>.
- [6] H. Whitney. 2-isomorphic graphs. *Amer. J. Math.*, 55:245-254, 1933.
- [7] D. K. Wagner. Connectivity in bicircular matroids. *J. Combin. Theory Ser. B*, 39:308-324, 1985.
- [8] T. Zaslavsky. Biased graphs. I. Bias, balance and gains. *J. Combin Theory Ser. B*, 47:32-52, 1989.
- [9] T. Zaslavsky. Biased graphs. II. The three matroids. *J. Combin Theory Ser. B*, 51: 46-72, 1991.